ISRAEL JOURNAL OF MATHEMATICS 97 (1997), 113–123

FOLIATIONS WITH LEAVES OF NONPOSITIVE CURVATURE*

BY

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ABSTRACT

We answer a question of Gromov ([G2]) in the codimension 1 case: if $\mathcal{F}$ is a codimension 1 foliation of a compact manifold M with leaves of negative curvature, then $\pi_1(M)$ has exponential growth. We also prove a result analogous to Zimmer's ([Z2]): if $\mathcal{F}$ is a codimension 1 foliation on a compact manifold with leaves of nonpositive curvature, and if $\pi_1(M)$ has subexponential growth, then almost every leaf is flat. We give a foliated version of the Hopf theorem on surfaces without conjugate points.

0. Introduction

A foliation is a manifold made out of leaves. The structure of the leaves affects the geometry and topology of the manifold and vice versa; the topology and geometry of the manifold also restrict the kind of leaves it could carry. A good example is Novikov's close leaf theorem: if a three dimensional manifold M carries a smooth foliation of codimension 1, and if $\pi_1(M)$ is finite, then the foliation has a closed leaf. A special kind of foliation comes from a faithful action of a connected Lie group G on a manifold M. Here all the leaves have the same sort of fabric. One of the main results of [G1] is the following: If G is a semisimple Lie group with no compact factor and finite fundamental group and if G acts real analytically on M preserving a pseudo-Riemannian metric, then $\widetilde{G}$ acts properly on a conull set of $\widetilde{M}$ (the universal covering). In

* Partially supported by NSF Grant #DMS 9403870.
Received January 8, 1994 and in revised form July 12, 1994

particular, $\pi_1(M) \geq G$ in the sense that there is a representation of $\pi_1(M)$ whose Zariski closure contains a subgroup locally isomorphic to G (this notion is due to Zimmer[Z3], for a geometric interpretation (called placement) see [G1]). See also [Z3] for stronger results along the line of Gromov's. In general, there are not many known topological restrictions on a G-manifold for a noncompact Lie group G except the obvious ones like the dimension restriction. This is due to our lack of general understanding of the usually nontrivial dynamics of the noncompact group action (for compact Lie groups there is no dynamics for the G-action but more is known about the topology. For example, there is a classical result saying that if a compact connected group G acts faithfully on a torus then G must be a torus). In [G2] and [DG], Gromov asks the following

QUESTION (Gromov): *Let $\mathcal{F}$ be a foliation with leaves of negative curvature on a compact manifold M. Assume $\mathcal{F}$ has a holonomy invariant measure. Does $\pi_1(M)$ have exponential growth?*

We give a partial answer to this question in the following

THEOREM 1: *Let $\mathcal{F}$ be a codimension 1 foliation of a compact manifold M with leaves of negative curvature; then $\pi_1(M)$ has exponential growth.*

Remarks: (1) Here we do not assume the existence of a holonomy invariant measure. We do not specify the regularity of the foliation and the leaf-wise Riemannian metric. In most cases, C^3 would be enough. (2) In [S], G. Stuck proved the following theorem. Theorem 1 can be viewed as a non-homogeneous generalization of his theorem.

THEOREM ([S]): *Let M be a smooth compact manifold, G a semisimple Lie group with finite center, and suppose G acts locally freely on M with orbits of codimension 1. Then $\pi_1(M)$ has exponential growth.*

(3) An open problem in dynamical system is: if a compact manifold M supports an Anosov flow, does $\pi_1(M)$ have exponential growth? The similarity between this problem and Gromov's is obvious: here the manifold M is foliated by the weak stable leaves of the Anosov flow and most of these leaves behave very much like a simply connected manifold of negative curvature. For example, their volume have exponential growth. If the weak stable foliation of the Anosov flow has codimension 1, then the answer is yes (see [PT]). (4) In the case of codimension 1 or in Gromov's case ([G1], where the foliation comes from a real-analytic action

of a semi-simple Lie group preserving a pseudo-Riemannian metric), the lifted foliation on the universal cover is proper. In the general high codimensional variable negative curvature case, we do not know whether the lifted foliation is still proper.

Recall that a transverse measure ω for a foliation $\mathcal{F}$ on M assigns to each compact transverse submanifold T to $\mathcal{F}$ a Borel measure ω_T. The measure is holonomy invariant if given two compact transversals T and T' such that $T' = \mathrm{hol}(T)$ under the holonomy map hol along $\mathcal{F}$, $\omega_T = \mathrm{hol}^*(\omega_{T'})$. Once such a measure is given, one can combine it with the Lebesgue measure along the leaves to obtain a measure on M. A general foliation need not possess any holonomy invariant measure. R. Zimmer has studied the orbit space of a foliation with negatively curved leaves from a measure theoretic point of view. One of his main results ([Z1] [Z2]) is that if $\mathcal{F}$ is a Riemannian measurable foliation with holonomy invariant measure and finite total volume, if almost every leaf is a complete simply connected manifold of nonpositive sectional curvature and moreover if $\mathcal{F}$ is amenable, then almost every leaf is flat. So as a corollary, if almost every leaf is a simply connected manifold of negative sectional curvature, then $\mathcal{F}$ is nonamenable.

In this paper, we prove the following theorem which is an analogue of the main result in [Z2].

THEOREM 2: *Let $\mathcal{F}$ be a codimension 1 foliation with leaves of nonpositive curvature on a compact manifold M. If $\pi_1(M)$ has subexponential growth, then almost all leaves are flat.*

It makes sense to raise the following question, which is a foliational analogue of the Hopf conjecture:

QUESTION: *If $\mathcal{F}$ is an amenable Riemannian measurable foliation with holonomy invariant measure of finite total volume and if almost every leaf of $\mathcal{F}$ is a complete simply connected manifold with no conjugate points, then almost every leaf is flat.*

One of the main ingredients in our proof of Theorem 1 is a simple observation that there is no vanishing cycle for a codimension 1 foliation with leaves of no conjugate points. An easy corollary of this observation is

COROLLARY 3: *There is no way to assign a C^2-leafwise Riemannian metric with no conjugate points along the Reeb foliation.*

We also prove the following

THEOREM 4: *Let $\mathcal{F}$ be an orientable foliation on a compact manifold M with 2-dimensional leaves having no conjugate points. If the Euler class of the tangent bundle of the foliation is zero in real homology and if all leaves have subexponential growth, then all leaves are flat.*

ACKNOWLEDGEMENT: I would like to express my sincere gratitude to Professor Zimmer for pointing out our ignorance in an earlier version of this paper and for other enlightening comments.

1. Proof of Theorem 1, the 3-dimensional case

The proof is best explained by the following case: $\dim M = 3$, $\dim \mathcal{F} = 2$. Let us first introduce some definitions.

Definition 1: If L is a leaf of a foliation $\mathcal{F}$ on M, then any non-trivial element in the kernel of the induced map $\pi_1(L) \to \pi_1(M)$ is called a **vanishing cycle**.

Definition 2: If $B_\gamma(x_0)$ denotes the ball of radius γ about x_0 in a complete Riemannian manifold M, set $h = \overline{\lim}_{r\to\infty} \frac{1}{r} \log(\mathrm{Vol}(B_r(x_0)))$. If $h = 0$, then M is said to have **subexponential growth**.

Many types of codimension-one foliations do not have vanishing cycles: for example, real analytic foliations, foliations determined by locally free Lie group actions, and Anosov foliations.

LEMMA 1$'$: *If $\mathcal{F}$ is a 2-dimensional foliation on a 3-dimensional manifold M with negatively curved leaves, then there is no vanishing cycle.*

Proof: If there is a vanishing cycle, then by Novikov's theorem, $\mathcal{F}$ contains a Reeb component. In particular, it contains a compact leaf T^2. This is impossible because of the negative curvature hypothesis.

LEMMA 2$'$: *Let Γ be a Fuchsian group. If $\mathbb{H}^2/\Gamma$ has subexponential growth, then Γ has exponential growth.*

Proof: Let $\widetilde{B}(x,R)$ be the ball in $\mathbb{H}^2$ of radius R, $\overline{B}(x,R)$ be the projection of $\widetilde{B}(x,R)$ to $\mathbb{H}^2/\Gamma$, $D_x = \{y \in \mathbb{H}^2 \,|\, d(y,x) \le d(y,\gamma x) \text{ for all } \gamma \in \Gamma\}$ be the Dirichlet domain of Γ at x and denote by x_0 the image of x in $\mathbb{H}^2/\Gamma$ under the canonical projection. Let $B(x_0,R)$ be the ball of radius R in $\mathbb{H}^2/\Gamma$. Then one has

$$\overline{B}(x,R) \subset B(x_0,R). \tag{1}$$

Now for any point $y \in \widetilde{B}(x,R)$. Let $y \in \gamma D_x = D_{\gamma x}$ for some $\gamma \in \Gamma$. By the definition of the Dirichlet domain $D_{\gamma x}$ we have

$$d(\gamma x, y) \le d(\gamma x, \gamma y) = d(x,y) \le R. \tag{2}$$

Therefore we get

$$d(\gamma x, x) \le d(\gamma x, y) + d(y,x) \le 2R. \tag{3}$$

Combining (1), (2), and (3), one has $\mathrm{Vol}(\widetilde{B}(x,R)) \le N(x,2R)\cdot \mathrm{Vol}(B(x_0,R))$, where $N(x,2R) \triangleq \#\{\gamma \in \Gamma \big| d(x,\gamma x) \le 2R\}$. Consequently we obtain

$$\lim_{R\to\infty} \frac{\log N(x,2R)}{2R} \ge \lim_{R\to\infty} \frac{\log V(\widetilde{B}(x,R))}{2R} - \overline{\lim_{R\to\infty}} \frac{\log \mathrm{Vol}(B(x_0,R))}{2R} \ge \frac{1}{2}. \qquad \blacksquare$$

Proof of Theorem 1 (the $\dim M = 3$ case)*:* If $\mathcal{F}$ has a leaf L of subexponential growth, then by Lemma 1′, $\pi_1(L)$ injects in $\pi_1(M)$. Also according to Lemma 2′, $\pi_1(L)$ has exponential growth, so does $\pi_1(M)$. If every leaf has exponential growth, since $\mathcal{F}$ has no vanishing cycle, by Novikov's result ([N]), there are no null-homotopic closed transversals to $\mathcal{F}$. Then Theorem 1 follows from the following result of Plante.

LEMMA ([P]): *Let $\mathcal{F}$ be a codimension one foliation on a compact manifold. If there exists a leaf of exponential growth which does not intersect any null-homotopic closed transversal, then $\pi_1(M)$ has exponential growth.*

2. Proof of Theorem 1, the general case

The proof of Theorem 1 is accomplished by Plante's lemma and the following two lemmas.

LEMMA 1: *Let $\mathcal{F}$ be a codimension one foliation with leaves of no conjugate points; then $\mathcal{F}$ has no vanishing cycle.*

LEMMA 2: *Let M be a complete Riemannian manifold with bounded geometry. If M has subexponential growth and $\widetilde{M}$ has exponential growth, then $\pi_1(M)$ has exponential growth.*

The proof of Lemma 2 goes almost word by word like the proof of Lemma 2′. We omit it here. Let us turn to the proof of Lemma 1.

Proof of Lemma 1: Suppose there is a vanishing cycle on the leaf L_{x_0}, $x_0 \in M$. Let $\tau = \tau(t)$, $0 \le t \le 1$, be a small segment of arc transverse to the foliation which contains $x_0 = \tau(0)$ as one endpoint. Assume also that for $x_t \triangleq \tau(t)$, $t > 0$, the deformation of the original loop from L_{x_0} to L_{x_t} is null-homotopic in L_{x_t}. Since the leaves of $\mathcal{F}$ have no conjugate points, there is a diffeomorphism $[0,1] \times \mathbb{R}^n \xrightarrow{\Phi} M$, where $n = \dim \mathcal{F}$, given by the following

$$\Phi(t,x) = \exp_{\tau(t)} f(t,x)$$

where $f(t,\cdot) = \mathbb{R}^n \to T_{\tau(t)}M$ is a family of linear isomorphisms continuous in t. For each $t \in [0,1]$, Φ_t: $\mathbb{R}^n \to L_{x(t)}$, $\Phi_t(x) \triangleq \Phi(t,x)$ is a covering map and, moreover, $\Phi_t(0) = \tau(t) = x_t$. Now the loop σ_0 representing the vanishing cycle in L_{x_0} is given by $s \to \Phi(0,\gamma(s))$, $0 \le s \le 1$, where $\gamma = [0,1] \to \mathbb{R}^n$ is a path in $\mathbb{R}^n$ and $\gamma(0) = 0$, $\gamma(1) \ne 0$ (because $\mathbb{R}^n \xrightarrow{\Phi_0} L_{x_0}$ is a covering projection and the loop $s \to \Phi_0(\gamma(s))$ is not null-homotopic in L_{x_0}). Let σ_t be the deformation of σ_0 from L_{x_0} to L_{x_t}, which is null-homotopic in L_{x_t}. Clearly σ_t can be taken in the form $s \longrightarrow \Phi_t(\gamma(s) * \delta_t(s))$, where δ_t: $[0,1] \longrightarrow \mathbb{R}^n$, $0 < t < 1$, is a continuous family of paths such that $\delta_t(0) = \gamma(1)$ and, as $t \to 0$, $\delta_t \to 0$ (by continuity).

Now for $t > 0$ the path $\gamma * \delta_t$ is closed in $\mathbb{R}^n$ (because $\mathbb{R}^n \xrightarrow{\Phi_t} L_{x_t}$ is a universal cover and σ_t is null-homotopic in L_{x_t}). Therefore, $\delta_t(1) = 0$ for all $t > 0$. By the continuous dependence of $\delta_t(1)$ on t we have $\delta_0(1) = 0$, which contradicts the fact that $\delta_0(s) = \gamma(1) \ne 0$ for $s \in [0,1]$.

Proof of Theorem 1: It can be achieved by an almost word by word translation of the proof in the 3-dimensional case.

3. Proof of Theorem 2

If $\mathcal{F}$ is a codimension 1 foliation with leaves of nonpositive curvature on a compact manifold M, and if $\pi_1(M)$ has subexponential growth, then by the arguments of Sections 2 and 3, all leaves L of $\mathcal{F}$ satisfy the following two properties:

(1) $\pi_1(L)$ has subexponential growth.
(2) The universal cover $\widetilde{L}$ of L has subexponential growth.

In particular, each leaf L has subexponential growth. By a theorem of Plante ([P]), such a leaf gives rise to an holonomy invariant measure ω of $\mathcal{F}$ which is supported on the closure of L with finite total volume. Another theorem of Plante ([P]) says that the support of any invariant measure of a codimension 1 foliation consists of leaves of polynomial growth. By a result of C. Series ([Se]), the foliation $\mathcal{F}$ is amenable with respect to ω. Therefore Zimmer's result ([Z2]) applies and Theorem 2 follows.

4. Proof of Theorem 4

Suppose $\mathcal{F}$ is a C^2 foliation on a compact manifold M equipped with a leafwise Riemannian metric with C^1-transverse smoothness. Let $S\mathcal{F}$ denote the bundle of unit vectors tangent to leaves of $\mathcal{F}$. The leaf-wise Riemannian metric g_L gives rise to a leafwise geodesic flow $g^t\colon \mathbb{R}\times S\mathcal{F}\to S\mathcal{F}$ defined by $g^t(v)=\dot{v}(t)$, $v\in SL$ (the unit tangent bundle of the leaf L), where $v(t)$ is the geodesic on L with initial velocity v: $\dot{v}(0)=v$. The dynamics of this flow contains some information of the foliation $\mathcal{F}$, but not all. If $\mathcal{F}$ has a holonomy invariant measure ω then one can combine ω with the Liouville measure m on each SL to obtain a measure $\overline{\omega}$ on $S\mathcal{F}$. Notice that the holonomy invariant measure ω is represented by a family of finite Borel measures defined on local transversals of the foilation which satisfy the property that they are invariant under the holonomy map induced by the foliation. It is enough to define the measure $\overline{\omega}$ locally in a flow box $T\times D^k$ of the foliation: one first integrates along the unit tangent bundle of the local leaves $t\times D^k$ with respect to the Liouville measure, then integrates along the transversal T with respect to the finite Borel measure representing ω. It is obvious that this gives rise to an invariant measure of the geodesic flow g^t, which we call again the Liouville measure of g^t with respect to ω. If L is a leaf without conjugate points, then for each $v\in SL$ there is a stable horosphere $H(v)$ perpendicular to v (see [EO] for a detailed discussion). Let $U(v)$ be the second fundamental form (if it exists) of $H(v)$. It is a semi-positive definite symmetric solution to the Riccati equation $\dot{U}+U^2+R=0$ where R is the curvature tensor of g_L. It was a classical result of L. Green that $U(v)$ is bounded and varies measurably in v. Therefore one can integrate on $S\mathcal{F}$ the Riccati equation with respect to $\overline{\omega}$ and obtain

$$\int_{S\mathcal{F}}(\mathrm{tr}\dot{U}+\mathrm{tr}U^2+\mathrm{tr}R)d\overline{\omega}=0.$$

Moreover $\int_{S\mathcal{F}}\mathrm{tr}\dot{U}d\overline{\omega}=0$ because $\overline{\omega}$ is g^t-invariant. Hence we have

$$\int_{S\mathcal{F}}\mathrm{tr}U^2d\overline{\omega}=-\frac{\omega_{n-1}}{n}\int_M S(x)d\omega(x)$$

where ω_{n-1} is the volume of unit ball in $\mathbb{R}^n$ and $S(x)$ is the scalar curvature of g_L on the leaf L_x. Therefore $U = 0$ and, by the Ricatti equation, the curvature tensor $R = 0$. Consequently we deduce the following

THEOREM 5: *Let $\mathcal{F}$ be a foliation on a manifold M with leafwise Riemannian metric without conjugate points. Then the total scalar curvature of $\mathcal{F}$ with respect to any finite holonomy invariant measure ω is nonpositive, and it vanishes if and only if ω-almost every leaf is flat.*

Now we can finish the proof of Theorem 4. We adapt the language of Phillips and Sullivan ([PS]). If L is a leaf of subexponential growth in $\mathcal{F}$, then there exists an exhausting sequence $B_0 = B_{r_0}(x) \subset B_1 = B_{r_1}(x) \subset \cdots \subset L$ which defines a holonomy invariant measure ω in the sense of [P]. If $\dim \mathcal{F} = 2$, then the scalar curvature coincides with sectional curvature K. Therefore

$$\int_M S(x) d\omega(x) = \lim_{k\to\infty} \frac{\int_{B_k} K(y) dy}{\text{area } (B_k)} \tag{1}$$

where dy is the Riemannian volume along L. Now the right hand side of (1), by [PS], gives rise to $\omega[E]$, where E is the Euler class of the tangent bundle of $\mathcal{F}$. If E is zero in real homology, then $\omega[E] = 0$ and therefore $\int_M S(x) d\omega(x) = 0$. Theorem 4 is proved.

5. General remarks on negatively curved foliations

A leaf L in a codimension one foliation $\mathcal{F}$ is called **resilient** if there is a transversal T and $x \in T \cap L$ such that x is a limit point of $T \cap L$ and there is a closed loop in L based at x, which generates a local holonomy γ which is a contraction in a neighborhood of x in T. If $0 < \gamma'(x) < 1$ then L is called **linearly resilient**. In [GLW], the authors gives a definition of foliation topological entropy $h(M/\mathcal{F})$ and prove the following

THEOREM ([GLW]): *Let $\mathcal{F}$ be a C^2-foliation of codimension one on a compact manifold M. Then $\mathcal{F}$ has a resilient leaf with linearly contracting holonomy if and only if $h(M/\mathcal{F}) > 0$.*

S. Hurder ([H]) proves that the topological entropy $h(g^t)$ of the geodesic flow on $S\mathcal{F}$ satisfies $h(g^t) > 0$ if $h(M/\mathcal{F}) > 0$. Let us recall also the following

THEOREM (Sacksteder): *Let $\mathcal{F}$ be a C^2-foliation of codimension-one on a compact manifold M. Then either $\mathcal{F}$ has an exceptional minimal set and therefore a linearly contracting resilient leaf or $\mathcal{F}$ supports a bundle like metric.*

Let us look at Theorem 1 from this new point of view. If $\mathcal{F}$ is a C^2-codimension one foliation which supports a bundle like metric, then the lift $\widetilde{F}$ of $\mathcal{F}$ to the universal cover $\widetilde{M}$ of M is proper and each leaf $\widetilde{L}$ of $\widetilde{F}$ is simply connected. If $\mathcal{F}$ supports a leafwise Riemannian metric of negative curvature, then each leaf $\widetilde{L}$ in $\widetilde{F}$ has exponential growth. Combining the properness of $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ it is easy to see that $\widetilde{M}$ must also have exponential growth and so is $\pi_1(M)$ (see [G1]). Moreover, if $\mathcal{F}$ has an exceptional minimal set and therefore a linearly contracting resilient leaf, then the theorem of [GLW] implies that $h(M/\mathcal{F}) > 0$ and therefore $h(g^t) > 0$ by the result of S. Hurder ([H]). On the other hand, Hurder introduces a notion of foliation entropy of the foliation geodesic flow $h(M/\mathcal{F}, g^t)$ and proves that $h(M/\mathcal{F})$ and $h(M/\mathcal{F}, g^t)$ either simultaneously vanish or are positive. He also introduces a notion of metric entropy $h_\nu(M/\mathcal{F}, g^t)$ for each g^t-invariant transverse measure of $S\mathcal{F}$ and then proves a variational principle.

$$h(M/\mathcal{F}, g^t) = \sup_{\nu \in \mathcal{M}(g^T, T)} h_\nu(M/\mathcal{F}, g^t)$$

where T is a complete transversal of $\mathcal{F}$ in M and $\mathcal{M}(g^t, T)$ is the set of transverse g^t-invariant probability ergodic measures. For each C^1-foliation $\mathcal{F}$ of codimension n there corresponds a linear holonomy cocycle $\Phi\colon \mathbb{R} \times S\mathcal{F} \to \mathrm{GL}(n, \mathbb{R})$, where $\Phi(t, v)$ is the linear holonomy along the geodesic $v(t)$ in terms of an orthonormal measurable framing of the normal bundle of $\mathcal{F}$. For each ergodic g^t-invariant measure transverse ν there exists a canonical set of transverse Lyapunov exponents $\lambda_1 < \lambda_2 \cdots < \lambda_s$ reflecting the infinitesimal hyperbolic *transverse* behavior of $\mathcal{F}$. Moreover, there is an entropy estimate of Margulis type:

$$h_\nu(M/\mathcal{F}, g^t) \leq \sum_{\lambda_i > 0} \lambda_i.$$

Now if $\mathcal{F}$ is a codimension 1 foliation with a linearly contracting resilient leaf, then $h(M/\mathcal{F}, g^t) > 0$ and, by the variational principle, $h_\nu(M/\mathcal{F}, g^t) > 0$ for some ergodic g^t-invariant probability transverse measure ν. Since codim $\mathcal{F} = 1$ there is only one transverse Lyapunov exponent λ_1 and it is positive $\lambda_1 > 0$. On the other hand, if $\mathcal{F}$ carries a leafwise metric of negative curvature, then g^t is also hyperoblic in the leaf direction. Therefore ν is actually a hyperbolic measure in the sense of [K1], [K2] and, by the results there, one has

$$\overline{\lim_{T\to\infty}} \frac{\log P_T(g^t)}{T} > 0$$

where $P_T(g^t)$ is the number of hyperbolic closed orbits with period $\leq T$.

If one knows that different closed geodesics represent different homotopy classes in $\pi_1(M)$, then Theorem 1 follows from the above argument easily. But this is in general not true (look at $S^1 \times M$ where M is a closed Riemannian manifold of negative curvature. Let σ be a closed geodesic in M. Then $s_1 \times \sigma$ and $s_2 \times \sigma$ are closed geodesics on different leaves which represent the same thing in $\pi_1(S^1 \times M)$).

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